could by themselves be financial successes. It is hoped that the day is coming when some enterprising American mathematical publishers will realize that leadership implies sacrifices, such as our big retail merchants have been making in their efforts to increase patronage. There is a great need for large mathematical treatises, encyclopedias, and other works of reference in the English language. Our mathematical development is to-day impeded more by the lack of such works than by any other one obstacle.

Monographs like the one under review are filling a great want in our literature, as they tend to create an interest in some of the rich fields of modern mathematics. Their brevity tends to encourage some who might be overawed by the extent of a comprehensive treatise. Professor Dickson seems to have been very successful in the choice of the material for the present monograph. Beginning with very simple examples from plane analytic geometry, the author furnishes a number of geometrical interpretations and applications of invariants and covariants in the first part of the book, which covers 29 pages.

Part II is devoted to the theory of invariants in non-symbolic notation and "treats of the algebraic properties of invariants and covariants, chiefly of binary forms; homogeneity, weight, annihilators, seminvariant leaders of covariants, law of reciprocity, fundamental systems, properties as functions of the roots, and production by means of differential operators." In Part III the symbolic notation of Aronhold and Clebsch is explained and illustrated by simple examples. Great care has been taken to present the matters in a clear manner and to avoid the difficulties which the beginner frequently encounters in this field.

A considerable number of exercises and illustrative examples are provided throughout the book, and there is a good index at the end of the volume. The author's deep mathematical insight combined with his special efforts to present matters in a clear manner have resulted in a monograph on this important subject which the beginners in this field, as well as those who have already made considerable progress therein, should read with unusual interest and profit.

G. A. MILLER.

SOLUTIONS OF PROBLEMS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

415. Proposed by C. N. SCHMALL, New York City.

Show that

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}.$$

I. SOLUTION BY HERMON L. SLOBIN, University of Minnesota.

Let us expand $f(x) = x$ as a Fourier cosine series,

$$f(x) = a_0/2 + a_1 \cos x + a_2 \cos 2x + \cdots.$$

Hence, we have

$$a_m = \frac{2}{\pi}\int_0^\pi x\cos mx dx = \frac{2}{m^2\pi}[(-1)^m - 1] \qquad m \neq 0,$$

and

$$a_0 = \frac{2}{\pi}\int_0^\pi x dx = \pi.$$

Hence,

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right].$$

Putting $x = 0$, we have

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$$

Note.—See also Byerly's *Fourier Series and Spherical Harmonics*, p. 40, where this special case is handled by means of a Fourier sine series. EDITORS.

II. SOLUTION BY THE PROPOSER.

Let

$$\Sigma_1 = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots;$$

and

$$\Sigma_2 = \frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \cdots.$$

Then

$$\Sigma_1 = \frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \cdots + \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \cdots$$

$$= \frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \cdots + \frac{1}{2^n}\left(\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots\right)$$

or,

$$\Sigma_1 = \Sigma_2 + \frac{1}{2^n}\Sigma_1, \qquad \therefore \Sigma_2 = \frac{2^n - 1}{2^n}\Sigma_1,$$

Whence, Σ_2 is known when Σ_1 is found. When $n = 2$, Σ_1 is known to be $\pi^2/6$; then

$$\Sigma_2 = \frac{3}{4}\frac{\pi^2}{6} = \frac{\pi^2}{8}.$$

Note.—The value $\Sigma_1 = \pi^2/6$ is deduced in Loney's *Trigonometry*, page 444, where also the value $\Sigma_2 = \pi^2/8$ is deduced by means of the development of $\cos\theta$ as an infinite product. Such a solution was given by A. M. HARDING, A. L. MCCARTY and the PROPOSER. Loney also uses the same process to find $1/1^4 + 1/3^4 + 1/5^4 + \cdots = \pi^4/96$ and $1/1^4 + 1/2^4 + 1/3^4 + \cdots = \pi^4/90$. EDITORS.

III. SOLUTION BY W. C. BRENKE, University of Nebraska.

Consider the series,

$$S(x) \equiv \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots,$$

which reduces to the given series when $x = 0$.

This series may be summed. Put

$$S_n(x) \equiv \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots + \frac{\cos (2n+1)x}{(2n+1)^2}.$$

Differentiating twice with respect to x, we have

$$S_n'(x) = -\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots + \frac{\sin (2n+1)x}{2n+1}\right],$$

and

$$S_n''(x) = -[\cos x + \cos 3x + \cdots + \cos (2n+1)x].$$

Multiplying both members of the last equation by $2 \sin x$ and expanding the products on the right by use of the formula, $2 \cos mx \sin x = \sin (m+1)x - \sin (m-1)x$, we have

$$2 \sin x S_n''(x) = -\sin (2n+2)x.$$

Hence,

$$S_n'(x) = -\int \frac{\sin (2n+2)x}{2 \sin x} dx + c_1.$$

Letting $n = \infty$, the integral vanishes as is easily seen on integrating by parts, and we have

$$S'(x) = c_1, \qquad 0 < x < \pi.$$

Hence

$$S(x) = c_1 x + c_2, \qquad 0 < x < \pi.$$

To determine the constants, note that $S'(\pi/2) = -(\pi/4)$, as follows at once from the series for $\tan^{-1} x$ when $x = 1$. Hence $c_1 = -(\pi/4)$. Also, since $S(\pi/2) = 0$ we get $c_2 = \pi^2/8$. Therefore,

$$S(x) = -\frac{\pi}{4}x + \frac{\pi^2}{8},$$

or

$$\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots = -\frac{\pi}{4}x + \frac{\pi^2}{8}, \qquad 0 < x < \pi.$$

Since the series converges uniformly for all values of x, we may put $x = 0$, which gives the required result.